An effective approach to Picard-Vessiot theory and the Jacobian Conjecture

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Abstract

In this paper we present a theorem concerning an equivalent statement of the Jacobian Conjecture in terms of Picard-Vessiot extensions. Our theorem completes the earlier work of T. Crespo and Z. Hajto which suggested an effective criterion for detecting polynomial automorphisms of affine spaces. We show a simplified criterion and give a bound on the number of wronskians determinants which we need to consider in order to check if a given polynomial mapping with non-zero constant Jacobian determinant is a polynomial automorphism. Our method is specially efficient with cubic homogeneous mappings introduced and studied in fundamental papers by H. Bass, E. Connell, D. Wright and L. Drużkowski.

1 Introduction

Let K denote an algebraically closed field of characteristic zero. Let $n > 0$ be a fixed integer and let $F = (F_1, \ldots, F_n) : K^n \to K^n$ be a polynomial mapping, i.e. $F_i \in K[X_1, \ldots, X_n]$ for $i = 1, \ldots, n$. We consider the Jacobian matrix $J_F = [\frac{\partial F_i}{\partial X_j}]_{1 \le i,j \le n}$. The Jacobian Conjecture states that if $\det(J_F)$ is a non-zero constant, then F has an inverse, which is also polynomial.

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The Jacobian Conjecture is one of Stephen Smale's problems (cf. [9], Problem 16), which are a list of important problems in mathematics for the twenty-first century. Originally the conjecture was formulated for $n = 2$ by O. Keller (cf. [7]). In 1982 H. Bass, E. Connell and D. Wright ([1]) showed that the general case follows from the case where $n \geq 2$ and $F = (X_1 + H_1, \ldots, X_n + H_n)$ and where each H_i is zero or homogeneous of degree 3. One year later L. Drużkowski ([5]) improved this result proving that if the Jacobian Conjecture is true for $n \geq 2$ and

$$F = (X_1 + (\sum_{j=1}^{n} a_{1j} X_j)^3, \ldots, X_n + (\sum_{j=1}^{n} a_{nj} X_j)^3), \tag{1}$$

then it holds in general. A polynomial mapping F of the form (1) with constant Jacobian is called a *Drużkowski mapping*. In 2001 Drużkowski [6] proved that in his reduction (1) it is enough to assume that the matrix $A = [a_{ij}]$ is nilpotent of degree 2, i.e. $A^2 = 0$.

In 2011 T. Crespo and Z. Hajto generalized a classical theorem of A. Campbell ([3]) by proving an equivalent statement of the Jacobian Conjecture in terms of Picard-Vessiot extensions (cf. [4], Theorem 2). Condition 4 in Theorem 2 in the work of Crespo and Hajto suggested an effective criterion for polynomial automorphisms of affine spaces. However, the effectivity is obstructed by the big number of generalized wronskians which have to be considered when the dimension of the affine space is growing. In this paper we present a simplified criterion for a polynomial automorphism of an affine space and prove that if the dimension of the space is n then it is enough to consider $\frac{1}{2}n^2(n+1) - n$ generalized wronskians. We believe that a deeper analysis of our algorithm may lead to the proof of the Jacobian Conjecture.

Let $(\mathcal{F}, \Delta_{\mathcal{F}})$ be a partial differential field with an algebraically closed field of constants $C_{\mathcal{F}}$ and $\Delta_{\mathcal{F}} = \{\partial_1, \ldots, \partial_m\}$. Let us consider a *linear partial differential system in matrix form* over $\mathcal{F}$, i.e. a system of equations of the form

$$\partial_i(Y) = A_i Y, \; i = 1, \ldots, m, \; A_i \in M_{n \times n}(\mathcal{F}). \tag{2}$$

A matrix $y \in GL_n(\mathcal{K})$, where $\mathcal{K}$ is a differential field extension of $\mathcal{F}$, is called a *fundamental matrix* for the system (2) if $\partial_i(y) = A_i y$ for $i = 1, \ldots, m$. We say that the system (2) is *integrable* if it has a fundamental matrix. A differential field extension $(\mathcal{G}, \Delta_{\mathcal{G}})$ of $(\mathcal{F}, \Delta_{\mathcal{F}})$ is a *Picard-Vessiot extension for the integrable system* (2) if the following holds: $C_{\mathcal{G}} = C_{\mathcal{F}}$, there exists a fundamental matrix $y = \{y_{ij}\} \in GL_n(\mathcal{G})$ and $\mathcal{G}$ is generated over $\mathcal{F}$ as a field by the entries of y, i.e. $\mathcal{G} = \mathcal{F}(\{y_{ij}\}_{1 \leq i,j, \leq n})$.

There is another definition of a Picard-Vessiot extension, formulated by Kolchin in [8]. Let $\mathcal{F}$ be a partial differential field of characteristic zero with $\Delta_{\mathcal{F}} = \{\partial_1, \ldots, \partial_m\}$ and algebraically closed field of constants $C_{\mathcal{F}}$. Let $\mathcal{G}$ be a differential field extension of $\mathcal{F}$. Let $Y_1, \ldots, Y_n$ denote indeterminates and let Θ denote the free commutative multiplicative semigroup generated by the elements of $\Delta_{\mathcal{F}}$. So $\theta \in \Theta$ is a differential operator of the form $\partial_1^{i_1} \ldots \partial_m^{i_m}$, where $i_1, \ldots, i_m \in \mathbb{Z}_+ \cup \{0\}$. Let us denote by $\Theta(k)$ the subset of Θ of the elements of order less than or equal to k. The determinant $\det(\theta_i y_j)_{1 \leq i,j \leq n}$ is called a *generalized wronskian determinant* and denoted by $W_{\theta_1, \ldots \theta_n}(y_1, \ldots, y_n)$. Kolchin called $\mathcal{G}$ a *Picard-Vessiot extension of* $\mathcal{F}$ if $C_{\mathcal{G}} = C_{\mathcal{F}}$ and there exist $\eta_1, \ldots, \eta_n \in \mathcal{G}$ linearly independent over $C_{\mathcal{F}}$ such that $\mathcal{G} = \mathcal{F}\langle \eta_1, \ldots, \eta_n \rangle$ and

$$\forall \theta_1, \ldots \theta_n \in \Theta(n) : \; \frac{W_{\theta_1, \ldots \theta_n}(\eta_1, \ldots, \eta_n)}{W_{\theta_{01}, \ldots \theta_{0n}}(\eta_1, \ldots, \eta_n)} \in \mathcal{F} \tag{3}$$

for some fixed $\theta_{01}, \ldots \theta_{0n}$ such that $W_{\theta_{01}, \ldots \theta_{0n}}(\eta_1, \ldots, \eta_n) \neq 0$.

Theorem 1 in [4] establishes the equivalence between the two definitions of Picard-Vessiot extension of partial differential fields presented above. Theorem 2 in [4], which is a differential version of the classical theorem of Campbell, gives an equivalent formulation of the Jacobian Conjecture. Let K be an algebraically closed field of characteristic zero and let $F = (F_1, \ldots, F_n) : K^n \to K^n$ be a polynomial map such that $\det(J_F) = c \in K \setminus \{0\}$. We can equipp $K(x_1, \ldots, x_n)$ with the Nambu derivations, i.e. derivations $\delta_1, \ldots, \delta_n$ given by

$$\begin{pmatrix} \delta_1 \\ \vdots \\ \delta_n \end{pmatrix} = (J_F^{-1})^T \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_1} \end{pmatrix}.$$

Observe that $K\langle F_1, \ldots, F_n \rangle = K(F_1, \ldots, F_n)$, i.e. $K(F_1, \ldots, F_n)$ is stable under $\delta_1, \ldots, \delta_n$. Moreover if $\det(J_F) = 1$, then $J_F^{-1} = [\delta_j x_i]_{1 \le i,j \le n}$.

The following theorem is a reformulation of theorems 1 and 2 in [4] in the form we will use in the sequel.

Theorem 1.1. *Let K and F be as above. Then the following conditions are equivalent:*

1) F is a polynomial automorphism

2) The matrix

$$W = \begin{bmatrix} 1 & x_1 & \ldots & x_n \\ 0 & \delta_1 x_1 & \ldots & \delta_1 x_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \delta_n x_1 & \ldots & \delta_n x_n \end{bmatrix}$$

is a fundamental matrix for an integrable system

$$\delta_k Y = A_k Y, \quad k = 0, \ldots, n,$$

where we are taking $\delta_0 = \mathrm{id}$, with $A_k \in M_{(n+1)\times(n+1)}(K(F_1, \ldots, F_n))$.

2 Wronskian criterion

Theorem 1.1 gives a method of checking if a given polynomial map F is a polynomial automorphism. If we denote $x_0 = 1$, then we may write $W = [\delta_i x_j]_{i,j=0,1,\ldots,n}$. Let us assume that $detW = 1$ (which is equivalent to $\det(J_F) = 1$). We are going to find $A_k = \delta_k W \cdot W^{-1}$, $k = 1, 2, \ldots, n$ in order to check if the entries of A_k's lie in $K(F_1, \ldots, F_n)$. We have that

$$\delta_k W = \begin{bmatrix} 0 & \delta_k x_1 & \ldots & \delta_k x_n \\ 0 & \delta_k \delta_1 x_1 & \ldots & \delta_k \delta_1 x_n \\ 0 & \delta_k \delta_2 x_1 & \ldots & \delta_k \delta_2 x_n \\ \ldots & & & \ldots \\ 0 & \delta_k \delta_i x_1 & \ldots & \delta_k \delta_i x_n \\ \ldots & & & \ldots \\ 0 & \delta_k \delta_n x_1 & \ldots & \delta_k \delta_n x_n \end{bmatrix} = \left[\omega_{ij}^k\right]_{i,j=0,1,\ldots,n}, \quad \text{where} \quad \omega_{ij}^k = \delta_k \delta_i x_j.$$

Let us find $W^{-1} = \left([D_{ij}]_{i,j=0,1,\ldots,n}\right)^T$, where D_{ij} denote the adjoint determinant of the element $\delta_i x_j$ of matrix W. We obtain that

$$D_{00} = 1 \quad \text{and} \quad \forall j \geq 1 : \, D_{0j} = 0,$$

$$D_{i0} = (-1)^{i+1+1} \begin{vmatrix} x_1 & \ldots & x_n \\ \delta_1 x_1 & \ldots & \delta_1 x_n \\ \ldots & \ldots & \ldots \\ \delta_{i-1} x_1 & \ldots & \delta_{i-1} x_n \\ \delta_{i+1} x_1 & \ldots & \delta_{i+1} x_n \\ \ldots & \ldots & \ldots \\ \delta_n x_1 & \ldots & \delta_n x_n \end{vmatrix} = (-1)^{i+0} \det\left([\delta_s x_t]_{s=0,1,\ldots,n;\, s\neq i;\, t=1,\ldots n}\right).$$

For $i, j \geq 1$ we get

$$D_{ij} = (-1)^{i+j} \begin{vmatrix} 1 & x_1 & \ldots & x_{j-1} & x_{j+1} & \ldots & x_n \\ 0 & \delta_1 x_1 & \ldots & \delta_1 x_{j-1} & \delta_1 x_{j+1} & \ldots & \delta_1 x_n \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \delta_{i-1} x_1 & \ldots & \delta_{i-1} x_{j-1} & \delta_{i-1} x_{j+1} & \ldots & \delta_{i-1} x_n \\ 0 & \delta_{i+1} x_1 & \ldots & \delta_{i+1} x_{j-1} & \delta_{i+1} x_{j+1} & \ldots & \delta_{i+1} x_n \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \delta_n x_1 & \ldots & \delta_n x_{j-1} & \delta_n x_{j+1} & \ldots & \delta_n x_n \end{vmatrix}.$$

So $D_{ij} = (-1)^{i+j} \det\left([\delta_s x_t]_{s,t=\mathbf{0},\mathbf{1},\ldots,\mathbf{n};\, s\neq i, t\neq j}\right) = (-1)^{i+j} \det\left([\delta_s x_t]_{s,t=\mathbf{1},\ldots,\mathbf{n};\, s\neq i, t\neq j}\right)$ and consequently $W^{-1} = [B_{ij}]_{i,j=0,1,\ldots,n}$, where

$$B_{ij} = (-1)^{i+j} D_{ji} = (-1)^{i+j} \begin{vmatrix} 1 & x_1 & \ldots & x_{i-1} & x_{i+1} & \ldots & x_n \\ 0 & \delta_1 x_1 & \ldots & \delta_1 x_{i-1} & \delta_1 x_{i+1} & \ldots & \delta_1 x_n \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \delta_{j-1} x_1 & \ldots & \delta_{j-1} x_{i-1} & \delta_{j-1} x_{i+1} & \ldots & \delta_{j-1} x_n \\ 0 & \delta_{j+1} x_1 & \ldots & \delta_{j+1} x_{i-1} & \delta_{j+1} x_{i+1} & \ldots & \delta_{j+1} x_n \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \delta_n x_1 & \ldots & \delta_n x_{i-1} & \delta_n x_{i+1} & \ldots & \delta_n x_n \end{vmatrix}.$$

We compute $A_k = \left[a_{ij}^k\right]_{i,j=0,1,\ldots,n} = \delta_k W \cdot W^{-1}$. We obtain $a_{i0}^k = 0$, i.e. the first column (i.e. the column indexed by j=0) is a zero column. Moreover for $j \geq 1$

$$a_{0j}^k = \sum_{r=1}^{n} \delta_k \delta_0 x_r \cdot B_{rj} = \sum_{r=1}^{n} \delta_k x_r \cdot B_{rj} = \delta_k x_1 \cdot (-1)^{1+j} \begin{vmatrix} \delta_1 x_2 & \ldots & \delta_1 x_n \\ \ldots & \ldots & \ldots \\ \delta_{j-1} x_2 & \ldots & \delta_{j-1} x_n \\ \delta_{j+1} x_2 & \ldots & \delta_{j+1} x_n \\ \ldots & \ldots & \ldots \\ \delta_n x_2 & \ldots & \delta_n x_n \end{vmatrix} + \ldots$$

$$\ldots + \delta_k x_n \cdot (-1)^{n+j} \begin{vmatrix} \delta_1 x_1 & \ldots & \delta_1 x_{n-1} \\ \ldots & \ldots & \ldots \\ \delta_{j-1} x_1 & \ldots & \delta_{j-1} x_{n-1} \\ \delta_{j+1} x_1 & \ldots & \delta_{j+1} x_{n-1} \\ \ldots & \ldots & \ldots \\ \delta_n x_1 & \ldots & \delta_n x_{n-1} \end{vmatrix} = \begin{vmatrix} \delta_1 x_1 & \delta_1 x_2 & \ldots & \delta_1 x_n \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{j-1} x_1 & \delta_{j-1} x_2 & \ldots & \delta_{j-1} x_n \\ \delta_k x_1 & \delta_k x_2 & \ldots & \delta_k x_n \\ \delta_{j+1} x_1 & \delta_{j+1} x_2 & \ldots & \delta_{j+1} x_n \\ \vdots & \vdots & \ddots & \vdots \\ \delta_n x_1 & \delta_n x_2 & \ldots & \delta_n x_n \end{vmatrix} = \begin{cases} 0 & ; \quad k \neq j \\ 1 & ; \quad k = j \end{cases} .$$

If $i, j \geq 1$, then $a_{ij}^k = \sum_{r=1}^n \delta_k \delta_i x_r \cdot B_{rj}$, this means we have

$$a_{ij}^k = \delta_k \delta_i x_1 \cdot (-1)^{1+j} \begin{vmatrix} \delta_1 x_2 & \ldots & \delta_1 x_n \\ \ldots & \ldots & \ldots \\ \delta_{j-1} x_2 & \ldots & \delta_{j-1} x_n \\ \delta_{j+1} x_2 & \ldots & \delta_{j+1} x_n \\ \ldots & \ldots & \ldots \\ \delta_n x_2 & \ldots & \delta_n x_n \end{vmatrix} + \ldots + \delta_k \delta_i x_n \cdot (-1)^{n+j} \begin{vmatrix} \delta_1 x_1 & \ldots & \delta_1 x_{n-1} \\ \ldots & \ldots & \ldots \\ \delta_{j-1} x_1 & \ldots & \delta_{j-1} x_{n-1} \\ \delta_{j+1} x_1 & \ldots & \delta_{j+1} x_{n-1} \\ \ldots & \ldots & \ldots \\ \delta_n x_1 & \ldots & \delta_n x_{n-1} \end{vmatrix} =$$

$$= \begin{vmatrix} \delta_1 x_1 & \delta_1 x_2 & \ldots & \delta_1 x_n \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{j-1} x_1 & \delta_{j-1} x_2 & \ldots & \delta_{j-1} x_n \\ \delta_k \delta_i x_1 & \delta_k \delta_i x_2 & \ldots & \delta_k \delta_i x_n \\ \delta_{j+1} x_1 & \delta_{j+1} x_2 & \ldots & \delta_{j+1} x_n \\ \vdots & \vdots & \ddots & \vdots \\ \delta_n x_1 & \delta_n x_2 & \ldots & \delta_n x_n \end{vmatrix}$$

The total number of considered determinants is $n(n+1)^2$, since for every δ_k we have $(n+1)^2$ of them and $k = 1, \ldots, n$. However for each A_k we can ignore the first row and the first column (i.e. the row and the column indexed by 0), since they consist of constant elements. Consequently, we can omit $2n+1$ of elements for each A_k. So there are n^3 wronskians left. We can easily observe that for every $j = 1, \ldots, n$ and for $k \neq i$ we have $a_{ij}^k = a_{kj}^i$. So we can omit $\binom{n}{2}$ of determinants for each j. Due to the lemma given below we can omit even more determinants.

Lemma 2.1. *Let $(K, ')$ be a differential field and let $A = [a_{ij}]_{i,j=1,\ldots,n} \in GL_n(K)$ be a nonsingular matrix with entries in K. Then*

$$(\det A)' = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}' =$$

$$= \begin{vmatrix} a'_{11} & a'_{12} & \ldots & a'_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a'_{21} & a'_{22} & \ldots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} + \ldots + \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{n1} & a'_{n2} & \ldots & a'_{nn} \end{vmatrix} =$$

$$= \begin{vmatrix} a'_{11} & a_{12} & \ldots & a_{1n} \\ a'_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a'_{12} & \ldots & a_{1n} \\ a_{21} & a'_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a'_{n2} & \ldots & a_{nn} \end{vmatrix} + \ldots + \begin{vmatrix} a_{11} & a_{12} & \ldots & a'_{1n} \\ a_{21} & a_{22} & \ldots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a'_{nn} \end{vmatrix}$$

Let us use lemma (2.1) to differentiate $\det W = 1$ with respect to each δ_k, $k = 1, \ldots, n$. We get that

$$\delta_1 \begin{vmatrix} \delta_1 x_1 & \ldots & \delta_1 x_n \\ \vdots & \ddots & \vdots \\ \delta_n x_1 & \ldots & \delta_n x_n \end{vmatrix} = \begin{vmatrix} \delta_1^2 x_1 & \ldots & \delta_1^2 x_n \\ \vdots & \ddots & \vdots \\ \delta_n x_1 & \ldots & \delta_n x_n \end{vmatrix} + \begin{vmatrix} \delta_1 x_1 & \ldots & \delta_1 x_n \\ \delta_1\delta_2 x_1 & \ldots & \delta_1\delta_2 x_n \\ \delta_3 x_1 & \ldots & \delta_3 x_n \\ \vdots & \ddots & \vdots \\ \delta_n x_1 & \ldots & \delta_n x_n \end{vmatrix} + \ldots + \begin{vmatrix} \delta_1 x_1 & \ldots & \delta_1 x_n \\ \vdots & \ddots & \vdots \\ \delta_{n-1} x_1 & \ldots & \delta_{n-1} x_n \\ \delta_1\delta_n x_1 & \ldots & \delta_1\delta_n x_n \end{vmatrix} = 0$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\delta_n \begin{vmatrix} \delta_1 x_1 & \ldots & \delta_1 x_n \\ \vdots & \ddots & \vdots \\ \delta_n x_1 & \ldots & \delta_n x_n \end{vmatrix} = \begin{vmatrix} \delta_n\delta_1 x_1 & \ldots & \delta_n\delta_1 x_n \\ \delta_2 x_1 & \ldots & \delta_2 x_n \\ \vdots & \ddots & \vdots \\ \delta_n x_1 & \ldots & \delta_n x_n \end{vmatrix} + \begin{vmatrix} \delta_1 x_1 & \ldots & \delta_1 x_n \\ \delta_n\delta_2 x_1 & \ldots & \delta_n\delta_2 x_n \\ \delta_3 x_1 & \ldots & \delta_3 x_n \\ \vdots & \ddots & \vdots \\ \delta_n x_1 & \ldots & \delta_n x_n \end{vmatrix} + \ldots + \begin{vmatrix} \delta_1 x_1 & \ldots & \delta_1 x_n \\ \vdots & \ddots & \vdots \\ \delta_{n-1} x_1 & \ldots & \delta_{n-1} x_n \\ \delta_n^2 x_1 & \ldots & \delta_n^2 x_n \end{vmatrix} = 0$$

Let us go back to considerations concerning the matrix A_k. We can use the equations given above to observe that for each $k = 1, \ldots, n$ we have $a_{11}^k + \ldots + a_{nn}^k = 0$. So for example

$$a_{kk}^k = -a_{11}^k - \ldots - a_{k-1,k-1}^k - a_{k+1,k+1}^k - \ldots - a_{nn}^k.$$

So we can omit n determinants more. Hence it is enough to check the following number of wronskian determinants

$$n^3 - n \cdot \binom{n}{2} - n = n^3 - \frac{1}{2}n^2(n-1) - n = \frac{1}{2}n^2(n+1) - n. \tag{4}$$

Let us observe that the number given in (4) is optimal, e.g. for $n = 2$, we have to consider 4 wronskian determinants.

3 Examples

In this section in order to explain how our criterion works for detecting polynomial automorphisms we shall present two explicit examples.

Example 1. Let us consider a well-known wild automorphism: the Nagata automorphism:

$F_1 = x_1 - 2x_2(x_3x_1 + x_2^2) - x_3(x_3x_1 + x_2^2)^2$
$F_2 = x_2 + x_3(x_3x_1 + x_2^2)$
$F_3 = x_3$

Using the computer algebra system Maple18 we first compute that $\det(J_F) = 1$ and next the entries of the matrices $\left[a_{ij}^k\right]_{i,j=1,2,3}$, for $k = 1, 2, 3$. We obtain the following results:

k=1:
$a_{11}^1 = -a_{22}^1 - a_{33}^1 = -2x_3^3(-2x_1x_3^3 - 2x_2^2x_3^2 - 2x_2x_3 + 1) = 4F_2F_3^4 - 2F_3^3$

$a_{12}^1 = -2x_3^5 = -2F_3^5$

$a_{13}^1 = 0$

$a_{21}^1 = (-4x_1x_3^4 - 4x_2^2x_3^3 - 4x_2x_3^2 + 2x_3)(-2x_1x_3^3 - 2x_2^2x_3^2 - 2x_2x_3 + 1) = 8F_2^2F_3^3 - 8F_2F_3^2 + 2F_3$

$a_{22}^1 = (-4x_1x_3^4 - 4x_2^2x_3^3 - 4x_2x_3^2 + 2x_3)x_3^2 = -4F_2F_3^4 + 2F_3^3$

$a_{23}^1 = 0$

$a_{31}^1 = -4x_1^3x_3^8 - 12x_1^2x_2^2x_3^7 - 12x_1x_2^4x_3^6 - 4x_2^6x_3^5 - 12x_1^2x_2x_3^6 - 24x_1x_2^3x_3^5 - 12x_2^5x_3^4 + 10x_1^2x_3^5 + 8x_1x_2^2x_3^4 - 2x_2^4x_3^3 + 16x_1x_2x_3^3 + 12x_2^3x_3^2 + 6x_2^2x_3 + 2x_2 = 4F_2^2F_3 + 4F_1F_2F_3^3 - 2F_1F_3^2 + 2F_2$

$a_{32}^1 = 2x_1^2x_3^7 + 4x_1x_2^2x_3^6 + 2x_2^4x_3^5 + 4x_1x_2x_3^5 + 4x_2^3x_3^4 - 4x_1x_3^4 - 2x_2^2x_3^3 - 2x_2x_3^2 - 2x_3 = -2F_1F_3^4 - 2F_2F_3^2 - 2F_3$

$a_{33}^1 = 0$

k=2:
$a_{11}^2 = a_{21}^1$
$a_{12}^2 = a_{22}^1$
$a_{13}^2 = a_{23}^1 = 0$

$a_{21}^2 = 16x_1^3x_3^8 + 48x_1^2x_2^2x_3^7 + 48x_1x_2^4x_3^6 + 16x_2^6x_3^5 + 48x_1^2x_2x_3^6 + 96x_1x_2^3x_3^5 + 48x_2^5x_3^4 - 24x_1^2x_3^5 + 24x_2^4x_3^3 - 48x_1x_2x_3^3 - 32x_2^3x_3^2 + 12x_1x_3^2 - 12x_2^2x_3 + 12x_2 = 16F_2^3F_3^2 - 24F_2^2F_3 + 12F_2$

$a_{22}^2 = -a_{11}^2 - a_{33}^2 = -8x_1^2x_3^7 - 16x_1x_2^2x_3^6 - 8x_2^4x_3^5 - 16x_1x_2x_3^5 - 16x_2^3x_3^4 + 8x_1x_3^4 + 8x_2x_3^2 - 2x_3 = -8F_2^2F_3^3 + 8F_2F_3^2 - 2F_3$

$a_{23}^2 = 0$

$a_{31}^2 = -8x_1^4x_3^9 - 32x_1^3x_2^2x_3^8 - 48x_1^2x_2^4x_3^7 - 32x_1x_2^6x_3^6 - 8x_2^8x_3^5 - 32x_1^3x_2x_3^7 - 96x_1^2x_2^3x_3^6 - 96x_1x_2^5x_3^5 - 32x_2^7x_3^4 + 24x_1^3x_3^6 + 24x_1^2x_2^2x_3^5 - 24x_1x_2^4x_3^4 - 24x_2^6x_3^3 + 64x_1^2x_2x_3^4 + 96x_1x_2^3x_3^3 + 32x_2^5x_3^2 - 10x_1^2x_3^3 + 36x_1x_2^2x_3^2 + 38x_2^4x_3 - 12x_1x_2x_3 + 4x_2^3 + 2x_1 = 8F_1F_2^2F_3^2 - 8F_1F_2F_3 + 8F_2^3 + 2F_1$

$a_{32}^2 = 4x_1^3x_3^8 + 12x_1^2x_2^2x_3^7 + 12x_1x_2^4x_3^6 + 4x_2^6x_3^5 + 12x_1^2x_2x_3^6 + 24x_1x_2^3x_3^5 + 12x_2^5x_3^4 - 10x_1^2x_3^5 - 8x_1x_2^2x_3^4 + 2x_2^4x_3^3 - 16x_1x_2x_3^3 - 12x_2^3x_3^2 - 6x_2^2x_3 - 2x_2 = -4F_1F_2F_3^3 + 2F_1F_3^2 - 4F_2^2F_3 - 2F_2$

$a_{33}^2 = 0$

k=3:
$a_{11}^3 = a_{31}^1$
$a_{12}^3 = a_{32}^1$
$a_{13}^3 = a_{33}^1 = 0$
$a_{21}^3 = a_{31}^2$
$a_{22}^3 = a_{32}^2$
$a_{23}^3 = a_{33}^2 = 0$

$a^3_{31} = 4x_1^5x_3^{10}+20x_1^4x_2^2x_3^9+40x_1^3x_2^4x_3^8+40x_1^2x_2^6x_3^7+20x_1x_2^8x_3^6+4x_2^{10}x_3^5+20x_1^4x_2x_3^8+80x_1^3x_2^3x_3^7+120x_1^2x_2^5x_3^6+80x_1x_2^7x_3^5+20x_2^9x_3^4-18x_1^4x_3^7-32x_1^3x_2^2x_3^6+12x_1^2x_2^4x_3^5+48x_1x_2^6x_3^4+22x_2^8x_3^3-64x_1^3x_2x_3^5-152x_1^2x_2^3x_3^4-112x_1x_2^5x_3^3-24x_2^7x_3^2+16x_1^3x_3^4-36x_1^2x_2^2x_3^3-100x_1x_2^4x_3^2-48x_2^6x_3+28x_1^2x_2x_3^2+8x_1x_2^3x_3-16x_2^5-2x_1^2x_3+8x_1x_2^2 = 4F_1^2F_2F_3^2-2F_1^2F_3+8F_1F_2^2$

$a^3_{32} = -2x_1^4x_3^9-8x_1^3x_2^2x_3^8-12x_1^2x_2^4x_3^7-8x_1x_2^6x_3^6-2x_2^8x_3^5-8x_1^3x_2x_3^7-24x_1^2x_2^3x_3^6-24x_1x_2^5x_3^5-8x_2^7x_3^4+8x_1^3x_3^6+12x_1^2x_2^2x_3^5-4x_2^6x_3^3+20x_1^2x_2x_3^4+32x_1x_2^3x_3^3+12x_2^5x_3^2-4x_1^2x_3^3+8x_1x_2^2x_3^2+10x_2^4x_3+4x_2^3-2x_1 = -2F_1^2F_3^3-4F_1F_2F_3-2F_1$

$a^3_{33} = -a^3_{11} - a^3_{22} = 0$

Example 2. Recently Dan Yan ([10]) has proved that the Jacobian Conjecture is true for the Drużkowski mappings in dimension $n \leq 9$, however only in the case when the matrix A (cf.(1)) has no zeros on its diagonal, and for general n and rank$A \leq 4$. Moreover Michiel de Bondt in his thesis [2] proved the validity of the Jacobian Conjecture for all Drużkowski mappings in dimension $n \leq 8$. Let us consider the following Drużkowski mapping in dimension 13.

$$\begin{array}{rcl}
F_1 &=& X_1 + \left(\frac{1}{6}X_4 + \frac{1}{6}X_5 - \frac{1}{3}X_6 - \frac{1}{6}X_7 - \frac{1}{6}X_8 + \frac{1}{3}X_9 + X_{13}\right)^3 \\
F_2 &=& X_2 + \left(\frac{1}{6}X_4 + \frac{1}{6}X_5 - \frac{1}{3}X_6 - \frac{1}{6}X_7 - \frac{1}{6}X_8 + \frac{1}{3}X_9 - X_{13}\right)^3 \\
F_3 &=& X_3 + \left(\frac{1}{6}X_4 + \frac{1}{6}X_5 - \frac{1}{3}X_6 - \frac{1}{6}X_7 - \frac{1}{6}X_8 + \frac{1}{3}X_9\right)^3 \\
F_4 &=& X_4 + \left(\frac{1}{6}X_1 + \frac{1}{6}X_2 - \frac{1}{3}X_3 + X_{12}\right)^3 \\
F_5 &=& X_5 + \left(\frac{1}{6}X_1 + \frac{1}{6}X_2 - \frac{1}{3}X_3 - X_{12}\right)^3 \\
F_6 &=& X_6 + \left(\frac{1}{6}X_1 + \frac{1}{6}X_2 - \frac{1}{3}X_3\right)^3 \\
F_7 &=& X_7 + \left(-\frac{1}{3}X_3 + \frac{1}{6}X_{10} + \frac{1}{6}X_{11} + X_{13}\right)^3 \\
F_8 &=& X_8 + \left(-\frac{1}{3}X_3 + \frac{1}{6}X_{10} + \frac{1}{6}X_{11} - X_{13}\right)^3 \\
F_9 &=& X_9 + \left(-\frac{1}{3}X_3 + \frac{1}{6}X_{10} + \frac{1}{6}X_{11}\right)^3 \\
F_{10} &=& X_{10} + \left(\frac{1}{6}X_4 + \frac{1}{6}X_5 - \frac{1}{3}X_6 - \frac{1}{6}X_7 - \frac{1}{6}X_8 + \frac{1}{3}X_9 + X_{12}\right)^3 \\
F_{11} &=& X_{11} + \left(\frac{1}{6}X_4 + \frac{1}{6}X_5 - \frac{1}{3}X_6 - \frac{1}{6}X_7 - \frac{1}{6}X_8 + \frac{1}{3}X_9 - X_{12}\right)^3 \\
F_{12} &=& X_{12} \\
F_{13} &=& X_{13}
\end{array}$$

In the above example $rank(A) = 5$. The computation of wronskians is involved, therefore we have presented it separately in our website *http://crypto.ii.uj.edu.pl/galois/*.

Remark 3.1. In his landmark paper [3] L.A. Campbell studied in fact general covering maps. Let us observe that our computational approach can be used as well for detecting Galois coverings. In this case we can not assume that the Jacobian determinant is a non-zero constant, however the computations are analogous.

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